ENUMERATIVE GEOMETRY: PAST, PRESENT, AND FUTURE

WERN JUIN GABRIEL ONG

Abstract. These are notes for a talk at Parker Glynn-Adey's Undergraduate seminar at University of Toronto – Scarborough Campus in June 2023. The talk is based on the paper [6] forthcoming at *Le Matematiche*. I will also allude to some forthcoming work [11]. Notes from the talk will be posted on my website.

1. What is Enumerative Geometry?

Enumerative geometry asks geometric questions of the form "how many" and expects answers in $\mathbb{N}$. Here are some examples of questions in enumerative geometry that have arisen over the history of mathematics. Classicaly the Greek mathematician Euclid posed the following question:

Question 1.1 (Euclid, 300 BCE)**.** *Given two points in the plane, how many lines pass through both points?*

Later, Apollonius, a mathematician in present day Anatolia in Turkey asked the following:

Question 1.2 (Apollonius, 200 BCE)**.** *Given three circles in the plane, how many circles are tangent to all three?*

And almost two millenia later, the Swiss mathematician Jakob Steiner posed the following problem:

Question 1.3 (Steiner, 1850s)**.** *Given five conic sections in the plane, how many conic sections are tangent to all five?*

We can see the similarities between these questions. We want to know how many geometric objects satisfy certain conditions. We will be thinking lots about questions of this form today.

2. Some Answers

You might have encountered some of the answers to these questions before. The first you likely encountered at some point over your years of schooling.

Theorem 2.1 (Euclid, 300 BCE)**.** *Given two points in the plane, there is one line passing through two points.*

You are less likely to have encountered the answers to the last two questions before, but there are still nice answers to them. Apollonius himself was able to show

Theorem 2.2 (Apollonius, 200 BCE)**.** *Given three circles in the plane, there are eight circles tangent to all three.*

and while Steiner himself first gave an incorrect proof, Michel Chasles later resolved the problem:

Theorem 2.3 (Chasles, 1850s)**.** *Given five conic sections in the plane, there are 3264 conic sections tangent to all five.*

It is perhaps surprising that there are well-defined answers that do not depend on the choice of circles. That is, under *mild assumptions*, there is *always* one line passing through *any* two points, there are *always* eight circles tangent to *any* three given circles, and there are *always* 3264 conics tangent to *any* five given conics. This surprising fact is what Hermann Schubert called "the principle of conservation of number". I'm being intentionally vague here, but I will return to these *mild assumptions* later in the talk.

3. The Geometry of Circles

See [7, Sec. 2.3]. A circle in the plane is determined by three parameters, the two coordinates for its center $(a, b) \in \mathbb{R}^2$ and radius r has equation

$$(x-a)^2 + (y-b)^2 - r^2 = 0.$$

We can thus think of the "space of circles" as three dimensional. *(Draw the standard 3-dimensional grid)* so here we can see how "walking around" in this space gives us a different circle. Starting at a given point, moving in the r-direction expands and shrinks the radius, and moving in the a and b-directions move around the center of the circle. Hopefully you now believe that there is a 3-dimensional space of circles. We will come back to this space of circles very soon.

4. Answers to Enumerative Problems

What makes this field of geometry so attractive, at least to me, is the existence of "nice" answers. But what does it mean to be a nice answer? We want the answer to a "how many" question should be a finite natural number, an answer of 0 or ∞ would be much less satisfying. So why do these questions have a finite natural number as an answer? Let us consider Apollonius' problem as a case study.

- The condition "the circle C is tangent to a given circle C'" is a 1-dimensional constraint on the 3-dimensional space of circles. In other words, given a circle C', there is a 2-dimensional family of circles C that are tangent to C'.
- Having a circle C being simultaneously tangent to three circles C_1, C_2, C_3 is equivalent to these three 2-dimensional subspaces of the 3-dimensional space of circles intersecting.
- Three surfaces (2-dimensional subspaces) of a 3-dimensional space meet in a 0-dimensional (finite) set.

Think of the "space of circles" we talked about before as $\mathbb{R}^3$ and the space of circles tangent to a given circle C' as a plane in $\mathbb{R}^3$. Intersecting 3 planes in $\mathbb{R}^3$ results in a point. While in the case of Apollonius' problem we are intersecting quadric cones in $\mathbb{P}^3$, the intuition still holds. A place where the details are all spelled out is in a recent paper of McKean [10].

This is why Apollonius problem asks us for the number of circles tangent to **three** circles. Given two circles, we would have a 1-dimensional family of circles tangent to both, and given four circles, there may be zero circles that are tangent to all four.

5. Circles Tangent to Three Conics

But let us consider a variation of Apollonius' problem. How many circles are tangent to three of "something else"? In particular, let us consider the following extension of Apollonius problem: instead of considering circles tangent to circles, let us consider circles tangent to conics.

Question 5.1 (Breiding, 2018)**.** *Given three conics in the plane, how many circles are tangent to all three?*

In 2005, Emiris and Tzoumas offer a partial answer [8].

Proposition 5.2 (Emiris and Tzoumas, 2005)**.** *Given three ellipses in the plane, there are at most 184 circles tangent to all three.*

And last year, we strengthened this result to the following.

Proposition 5.3 (Prop. 2.1 of [6])**.** *Given three general conics in the plane, there are exactly 184 circles tangent to all three.*

Unlike the proof of Euclid's theorem and Apollonius' theorem, the proof here requires some more advanced machinery so I won't be getting into it, but I hope the statement of the result is of intrinsic interest.

6. A White Lie

Okay, now it is time for a confession. All the pictures I drew for you were drawn over $\mathbb{R}$ but all of the statements we have discussed above are only true over $\mathbb{C}$, since these statements boil down to counting the number of solutions to (multivariate) polynomials. That is, we need to work over an algebraically closed field, where the fundamental theorem of algebra holds.

But a polynomial of degree n over $\mathbb{R}$ need not have n roots.

So an interesting question to ask is what versions of the theorems we discussed before hold over $\mathbb{R}$?

On one hand, these questions are of intrinsic mathematical interest, while on the other are closely tied to questions in applied mathematics such as in robotics and physics, among others.

7. Real Enumeartive Problems

But pictures are nice, so let us think about real enumerative geometry. We can rephrase the question we first started with. Real enumerative geometry asks questions about real geometric objects of the form "how many" and expects answers in $\mathbb{N}$. We can even ask the same questions as before:

Question 7.1 (*a la* Euclid, 300 BCE)**.** *Given two real points in the plane, how many real lines pass through both points?*

Question 7.2 (*a la* Apollonius, 200 BCE)**.** *Given three real circles in the real plane, how many real circles are tangent to all three?*

Question 7.3 (*a la* Steiner, 1850s)**.** *Given five real conic sections in the real plane, how many real conic sections are tangent to all five?*

But the answers to these questions over the real numbers will be much more nuanced.

8. What can be done over the real numbers?

Why is this so? Over $\mathbb{R}$, we lose "invariance of number". We no longer have well-defined solution counts. For example the answer to the question "how many solutions does a real quadratic polynomial have?" is 0 **or** 2. But let us tackle our questions in turn. In the case of Euclid's theorem, this turns out not to be an issue, and the previous result applies.

Theorem 8.1 (Euclid, 300 BCE)**.** *Given two real points in the real plane, there is exactly one real line passing through them both.*

However, in the case of Apollonius' problem, we start to see this phenomenon of the loss of invariance of number.

Theorem 8.2 (Pedoe, 1970; [12]). *Given three real circles in the real plane, there can be* $\{0, 1, 2, 3, 4, 5, 6, 8\}$ *real circles tangent to all three.*

Over the real numbers we are not guaranteed the 8 solutions we saw before over the complex numbers. In fact, we can get every number between 0 and 8 except 7. That is, there is a configuration of three real cirles such that there are 0, 1, 2, 3, 4, 5, 6, 8 real circles tangent to all three.

9. Steiner's Problem – Real-ly

Finally, let us consider Steiner's problem over $\mathbb{R}$. Recall the question asks for the number of real conics mutually tangent to five real conics in the real plane. We have a partial answer to this question.

Theorem 9.1 (Ronga, et. al.; [15]). *There exists a configuration of five real conics in the real plane with 3264 real conics tangent to all five.*

Note that this result does not give us a full understanding of the real version of Steiner's problem. In particular, it does not state what numbers of real tangent conics are possible. This leads us to the following question.

Question 9.2. *What are the possible numbers of the number of real conics tangent to five given real conics in the real plane?*

Ideally, we would come up with a list of numbers of possible real solution counts. If you are interested in this problem, I would encourage you to read my notes where I talk about possible approaches to this problem.

Remark. *A computational approach might be taken following section 3 of [6] while a theoretical approach might use* $\mathbb{A}^1$*-homotopy theory. For this, see [16] and [9] for the basics of the theory and [2] for a treatment of Steiner's problem in the* $\mathbb{A}^1$*-homotopy theoretic context.*

10. Real Circles Tangent to Three Conics

So real enumerative goemetry is complicated. What can we say about the number of real circles tangent to three conics?

Question 10.1 (Breiding, 2018). *Given three conics in the plane, how many circles are tangent to all three? Of these 184, how many can be real?*

The number of complex solutions gives an upper bound to the number of real solutions. But is this upper bound attained? In the case of the real version of Steiner's conic problem, the answer is yes: there are 3264 complex solutions and the case of 3264 real solutions is possible too. In the case of real circles tangent to three conics, we were able to show the following:

Theorem 10.2 (Thm. 1.3 and 2.2 of [6]). *There exists a configuration of three real conics with 136 real circles tangent to all three.*

and were in fact able to prove a partial classification of the possible solutions:

Theorem 10.3 (Thm. 3.1 of [6]). *For every* $k \in \{0, 2, 4, \ldots, 136\}$*, there exists a configuration of three real conics with* k *real circles tangent to all three.*

However, this is incomplete as compared to [12]. We don't know what happens between 136 – what we have shown can be attained over $\mathbb{R}$ – and 184, the complex count. This leads us to the following conjecture.

Conjecture 10.4 (Conj. 1.4 of [6]). *There are at most 136 real circles tangent to three conics.*

11. Classifying Solutions

A conic is determined by six coefficients

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

and three conics are thus determined by 18 coefficients. Fixing a vector of length 18 is equivalent to fixing a configuration of three plane conics. Given a length 18 vector, can we tell how many real conics are tangent to the configuration of three conics it defines?

This is a pattern recognition problem, ie. the pattern between the 18 numbers tells us the number of real solutions. This is a task primed for the application of machine learning.

And in fact we show that machine learning works for this task: that is, there is a machine learning model that given a length 18 vector, correctly predicts the number of real circles tangent to the configuration of three conics defined by the length 18 vector. The machine learning model attains a classification accuracy of 95%.

12. Algebraic Geometry and Machine Learning

But how generalizable is the phenomenon here? Did we get lucky or can machine learning be used in geometry more broadly? Recently, we got the referee's report for the paper back

> I am skeptical on the general feasibility of the machine learning approach for classification problems in (real) algebraic geometry, but this is rather a philosophical comment and it is clear that the question by itself is interesting ...

So how else can we use machine learning in (real) geometry?

13. Discriminants

One way we can use machine learning in geometry is to understand the discriminant. Let us recall the (informal) definition.

Definition 13.1 (Discriminant). *The discriminant is a subspace of dimension $n-1$ in the n-dimensional space of coefficients that tells us when a polynomial has a root with multiplicity.*

We have already seen one example of this in high school: the quadratic discriminant $b^2 - 4ac$ tells us if a degree polynomial with real coefficients $ax^2 + bx + c \in \mathbb{R}[x]$ has 0 or 2 real roots.

In forthcoming work with Anna Seigal, we show that we can use machine learning to learn the quadratic discrminant to high accuracy. So, to the referee, there are at least two cases where machine learning can help us understand real geometry. Okay, I'm almost done.

14. Geometry, Computation, and the Past

Since antiquity, mathematicians have been interested in questions of enumerative geometry. On the left, we have the five Platonic solids, polyhedra where all sides are the same shape. And on the right, we have a clay tablet from present day Iraq recording the computation of Pythagorean triples.

15. Geometry, Computation, and the Present

Today, we recognize how these classical questions are closely tied into questions that not only are of intrinsic interest to mathematicians but also drawn from domains of applications. For example, comptuation can be used to solve polynomial systems that arise in robotics, number theory, and statistics.

These connections are enabed by a wide array of software that has been developed for solving mathematical problems such as

- Computational commutative algebra and algebraic geometry in `Macaulay2`
- Computational number theory in `SageMath`
- Numerical algebraic geometry in `HomotopyContinuation.jl`
- Computational group theory in `GAP`
- And many more...

16. Geometry, Computation, and the Future

But this is only the beginning of the fruitful interactions between geometry and computation. Just to highlight some work being done in the areas I discussed before:

- Sameer Agarwal, Tim Duff, Max Leiblich, Rekha Thomas, Paul Breiding, Felix Rydell, Elima Shehu, and Angelica Torres, among others, are working on the connections between geometry and computer vision [1, 5].
- In 1900, Hilbert first asked in his Problem 10 if there existed an algorithm to decide if there existed integer solutions to polynomials with integer coefficients. Yuri Matiyasevich building on work of Julia Robinson, Martin Davis, and Hilary Putnam showed that no such algorithm exists. More recent work of Bjorn Poonen attempts to treat this question in the setting of rings of algebraic integers [14, 13].
- In the last few years, Marie Brandenberg, Christian Hasse, Ben Hollering, Irem Portakal, and Ngoc Tran hac been investigating the applications of geometry to auctions and game theory [3, 4].

This is only a sampling of what's out there. There are many more questions and connections for you to go out there and explore.

References

[1] Sameer Agarwal et al. "An Atlas for the Pinhole Camera". In: *Foundations of Computational Mathematics* (Sept. 2022). DOI: 10.1007/s10208-022-09592-6. URL: https://doi.org/10.1007%2Fs10208-022-09592-6.

[2] Tom Bachmann and Kirsten Wickelgren. *On Quadratically Enriched Excess and Residual Intersections*. 2021. arXiv: 2112.05960 [math.AG].

[3] Marie-Charlotte Brandenburg, Christian Haase, and Ngoc Mai Tran. *Competitive equilibrium always exists for combinatorial auctions with graphical pricing schemes*. 2021. arXiv: 2107.08813 [math.CO].

[4] Marie-Charlotte Brandenburg, Benjamin Hollering, and Irem Portakal. *Combinatorics of Correlated Equilibria*. 2022. arXiv: 2209.13938 [math.CO].

[5] Paul Breiding et al. *Line Multiview Varieties*. 2022. arXiv: 2203.01694 [math.AG].

[6] Paul Breiding et al. *Real circles tangent to 3 conics*. 2022. URL: https://arxiv.org/abs/2211.06876.

[7] David Eisenbud and Joe Harris. *3264 & All That: Intersection Theory in Algebraic Geometry*. Cambridge University Press, 2016.

[8] Ioannis Z. Emiris and George M. Tzoumas. "Algebraic study of the Apollonius circle of three ellipses". In: *In Proc. Europ. Works. Comp. Geom.* 2005, pp. 147–150.

[9] Jesse Leo Kass and Kirsten Wickelgren. "An arithmetic count of the lines on a smooth cubic surface". In: *Compositio Mathematica* 157.4 (2021), pp. 677–709. DOI: 10.1112/S0010437X20007691.

[10] Stephen McKean. *Circles of Apollonius two ways*. 2022. arXiv: 2210.13288 [math.AG].

[11] Wern Juin Gabriel Ong and Anna Seigal. "Neural Networks and Discriminant Loci of Parameterized Polynomial Systems: First Examples". In: *Forthcoming* (2023).

[12] Daniel Pedoe. "The Missing Seventh Circle". In: *Elemente der Mathematik* 25 (1970).

[13] Bjorn Poonen. "Hilbert's Tenth Problem and Mazur's Conjecture for large subrings of $\mathbb{Q}$". In: *Journal of the American Mathematical Society* 16.4 (July 2003), pp. 981–990. DOI: 10.1090/s0894-0347-03-00433-8. URL: https://doi.org/10.1090%2Fs0894-0347-03-00433-8.

[14] Bjorn Poonen. "Using Elliptic Curves of Rank One towards the Undecidability of Hilbert's Tenth Problem over Rings of Algebraic Integers". In: *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2002, pp. 33–42. DOI: 10.1007/3-540-45455-1_4. URL: https://doi.org/10.1007%2F3-540-45455-1_4.

[15] Felice Ronga, Alberto Tognoli, and Thierry Vust. "The number of conics tangent to five given conics: the real case." eng. In: *Revista Matemática de la Universidad Complutense de Madrid* 10.2 (1997), pp. 391–421. URL: http://eudml.org/doc/44276.

[16] Kirsten Wickelgren. $\mathbb{A}^1$ *Enumerative Geometry*. 2019. URL: https://swc-math.github.io/aws/2019/2019WickelgrenNotes.pdf.

Bowdoin College, Brunswick, Maine 04011
Email address: gong@bowdoin.edu
URL: https://wgabrielong.github.io/